Equivalence of Representation

G - group

$\varphi : G \to GL(V)$, $\psi : G \to GL(W)$
representations

an **equivalence** from φ to ψ is a linear isomorphism $T : V \to W$ such that

$$\psi_g = T \circ \varphi_g \circ T^{-1} \quad \forall g \in G$$

Say that two G-representations φ and ψ are **equivalent** if $\exists$ an equivalence.

Write "$\varphi \sim \psi$".

$\to$ equivalence is an equivalence relation on the collection of representations of G

Example: $G = \mathbb{Z}_n, \quad n \geq 1$

$\varphi, \psi : \mathbb{Z}_n \longrightarrow GL_2(\mathbb{C})$

$$\varphi_{[k]} = \begin{bmatrix} \cos 2\pi k/n & -\sin 2\pi k/n \\ \sin 2\pi k/n & \cos 2\pi k/n \end{bmatrix}$$

$\varphi \sim \psi$

$$\psi_{[k]} = \begin{bmatrix} e^{2\pi i k/n} & 0 \\ 0 & e^{-2\pi i k/n} \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}$.

Check: $\psi_{[k]} = A \varphi_{[k]} A^{-1}$

so
$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{L_A} & \mathbb{C}^2 \\ \parallel & & \parallel \\ V & & W \end{array}$$
is an equivalence from φ to ψ

Given:

$\varphi : G \to GL(V)$, $\quad \dim V = n < \infty$

Choose basis $B = \{v_1, \dots, v_n\}$ of V

Define: $\quad T : V \to \mathbb{C}^n$ iso

such that $\quad T(v_k) = e_k$.

$T(x_1 v_1 + \dots + x_n v_n) = (x_1, \dots, x_n)$, $\quad x_k \in \mathbb{C}$

Define: $\quad \psi_g := T \varphi_g T^{-1} : \mathbb{C}^n \to \mathbb{C}^n$

so $\quad \psi_g \in GL_n(\mathbb{C}) \quad \forall\, g \in G$

$\Rightarrow \quad \psi : G \to GL_n(\mathbb{C})$ is a matrix rep.

and $\quad T : V \to \mathbb{C}^n$ is an equivalence

$\varphi \sim \psi$

$\Rightarrow$ all finite dimensional reps of G are equivalent to matrix representations

Goal: classify representations of G up to equivalence $\quad [F = \mathbb{C},\ \dim < \infty]$